Analysis 1, Summer 2023

List 5

Increasing, decreasing, critical points, absolute extremes

113. (a) For what value(s) of x does $x^3 - 18x^2 = 0$? $\boxed{x = 0, x = 18}$

   (b) For what value(s) of x does $3x^2 - 36x = 0$? $\boxed{x = 0, x = 12}$

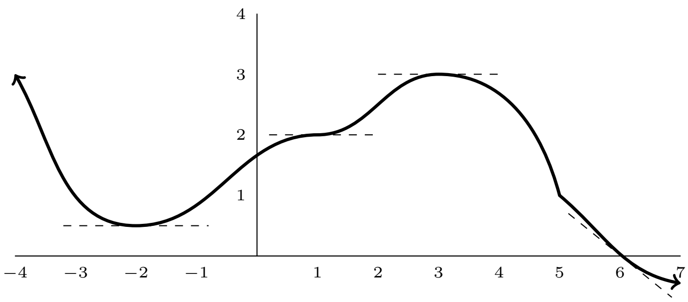
   (c) For what value(s) of x does $6x - 36 = 0$? $\boxed{x = 6}$

> A number c is a **critical point** of $f(x)$ if either $f'(c)$ does not exist or $f'(c) = 0$.
>
> If $f'(a) > 0$ then f is **increasing** at $x = a$.
> If $f'(a) < 0$ then f is **decreasing** at $x = a$.

114. What are the critical points of $x^3 - 18x^2$? $\boxed{x = 0, x = 12}$

115. Find all the critical points of $8x^5 - 57x^4 - 24x^3 + 9$. $\boxed{0, 6, \frac{-3}{10}}$

116. List all the critical points of the function graphed below (portions of its tangent lines at $x = -2$, $x = 1$, $x = 3$, and $x = 6$ are shown as dashed lines).



Critical points are $\boxed{2, 1, 4, 5}$.

117. Is the function

$$f(x) = x^8 - 6x^3 + 29x - 12$$

increasing, decreasing, or neither when $x = -1$? $\boxed{\text{increasing}}$

118. (a) On what (possibly infinite) interval or intervals is $2x^3 - 3x^2 - 12x$ decreasing?

   $\boxed{-1 < x < 2}$, which is $\boxed{(-1, 2)}$ in interval notation.

   (b) On what (possibly infinite) interval or intervals is $2x^3 - 3x^2 - 12x$ increasing?

   $\boxed{x < -1 \text{ or } x > 2}$, which is $\boxed{(-\infty, -1) \cup (2, \infty)}$ in interval notation.

119. List all critical points of $f(x) = \frac{3}{4}x^4 - 7x^3 + 15x^2$ in the interval $[-3, 3]$.

   $f'(c) = 0$ for $c = 0, 2, 5$, but only $\boxed{0 \text{ and } 2}$ are in the interval $[-3, 3]$.

120. For each graph below, is there a critical point at $x = 0$?

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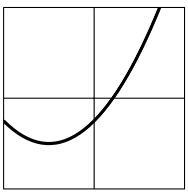

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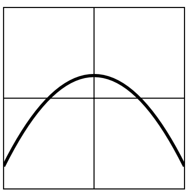

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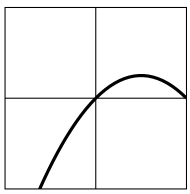
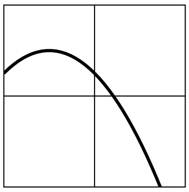
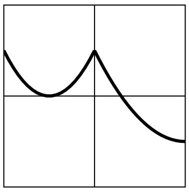
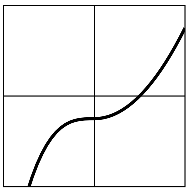


(d)  (e)  (f) 

(a) No, (b) Yes, (c) No, (d) No, (e) Yes, (f) Yes

121. The derivative of

$$f(x) = \frac{4x+1}{3x^2-12} \qquad \text{is} \qquad f'(x) = \frac{-4x^2-2x-16}{3x^4-24x^2+48}.$$

Using this, find all the critical points of $f(x)$.

$-4x^2 - 2x - 16 = 0$ has no real solutions, but $3x^4 - 24x^2 + 48 = 0$ when $\boxed{x = 2,\ x = -2}$, so f' does not exist at those points.

The derivative of $\sin(x)$ is $\cos(x)$. The derivative of $\cos(x)$ is $-\sin(x)$. In symbols,

$$\frac{\mathrm{d}}{\mathrm{d}x}\big[\sin(x)\big] = \cos(x) \qquad \text{and} \qquad \frac{\mathrm{d}}{\mathrm{d}x}\big[\cos(x)\big] = -\sin(x).$$

122. Give the derivative of $5\sin(x) + \frac{2}{3}\cos(x) - x^3 + 9$. $\boxed{5\cos(x) - \frac{2}{3}\cos(x) - 3x^2}$

123. Give an equation for the tangent line to $y = \sin(x)$ at $x = \frac{\pi}{3}$. $\boxed{y = \frac{\sqrt{3}}{2} + \frac{1}{2}(x - \frac{\pi}{3})}$

124. Find all the critical points of

(a) $f(x) = x^2 - \cos(x)$. $f' = 2x + \sin(x) = 0$ means $\sin(x) = -2x$, which is true only for $\boxed{x = 0}$.

(b) $f(x) = x + 2\cos(x)$. $f' = 1 - 2\sin(x) = 0$ means $\sin(x) = \frac{1}{2}$, which is true for $\boxed{x = \frac{1}{6}\pi + 2k\pi \text{ and } x = \frac{5}{6}\pi + 2k\pi}$, where k can be any integer.

(c) $f(x) = 2x + \cos(x)$. $f' = 2 - \sin(x) = 0$ when $\sin(x) = 2$, but this never happens for real values of x. So this function has $\boxed{\text{no critical points}}$.

(d) $f(x) = x^2 + x - \sin(x)$. $f' = 2x + 1 - \cos(x) = 0$ when the curves $y = \cos(x)$ and $y = 2x + 1$ intersect. This happens only at $\boxed{x = 0}$.

☆ (e) $f(x) = x^2 + x + \cos(x)$. $f' = 2x + 1 - \sin(x) = 0$ when the curves $y = \sin(x)$ and $y = 2x+1$ intersect. There is one point where this occurs, but there is no nice (technically, "closed form") formula for this value. It is approximately $\boxed{x = -0.335418}$.

To find the absolute extremes of a fn. on a closed, bounded interval:
① Find the critical points of f but *ignore critical points outside the interval.*
② Compute the value of f at the critical points *and* the endpoints of the interval.
③ The point(s) from ② with the largest f-value are absolute max, and point(s) with the smallest (i.e., most negative) f-value are absolute min.

125. On the interval $[-6, 3]$, find the absolute extremes of

$$2x^3 - 21x^2 + 60x - 20.$$

$f' = 6x^2 - 42x + 60$, so CP at 2 and 5. Ignore $x = 5$ bc it's not in $[-6, 3]$. Endpoints at -6 and 3.

x	f	
-6	-1568	abs. minimum
2	32	abs. maximum
3	25	(neither)

126. Find the absolute extremes of

$$x^4 - 4x^3 + 4x^2 - 14$$

on the interval $[-3, 3]$.

min ($f = -14$) at $x = 0$ and $x = 2$, max ($f = 661$) at $x = -3$ Note that this function has two absolute minima (the same minimum f-value at two x-values).

127. Find the absolute extremes of $x + 2\cos(x)$ with $0 \leq x \leq 2\pi$.

See **Task 124(b)** for the critical points. $x = \frac{\pi}{6}$ is the only CP in the interval $[0, 2\pi]$. Including the endpoints, we have

x	f	
0	2	abs. minimum
$\pi/6$	$\frac{\pi}{6} + \sqrt{3} \approx 2.26$	(neither)
2π	$2 + 2\pi \approx 8.28$	abs. maximum

128. Find the absolute minimum and absolute maximum of

$$f(x) = \tfrac{3}{4}x^4 - 7x^3 + 15x^2$$

with $|x| \leq 3$.

See **Task 119** for the CP: 0 and 2.

x	f	
-3	384.75	abs. maximum
0	0	abs. maximum
2	16	(neither)
3	6.75	(neither)

129. (a) Does the function $\dfrac{x-5}{x+2}$ have an absolute maximum on the interval $[-8, 4]$?

no (Note: the EVT does not apply because the function is not continuous (in fact, not defined) at $x = -2$.)

(b) Does the function $\dfrac{x-5}{\cos(x)+2}$ have an absolute maximum on $[-8, 4]$?

yes by the Extreme Value Theorem

130. Give the derivative of each of the following:

(a) $\frac{1}{2}x^4 + 4\sin(x) \quad \rightarrow \quad \boxed{2x^3 + 4\cos(x)}$

(b) $2x^2 + 4\cos(x) \quad \rightarrow \quad \boxed{4x - 4\sin(x)}$

(c) $4x - 4\sin(x) \quad \rightarrow \quad \boxed{4 - 4\cos(x)}$

(d) $4 - 4\cos(x) \quad \rightarrow \quad \boxed{4\sin(x)}$

(e) $4\sin(x) \quad \rightarrow \quad \boxed{4\cos(x)}$

131. A car drives in a straight line for 10 hours with its position after t hours being $24t^2 - 2t^3$ kilometers from its initial position. How far away is the farthest point the car reaches in 10 hours, and when does this occur? $\boxed{512 \text{ km away at } t = 8 \text{ hr}}$

Product Rule: $(fg)' = fg' + f'g$, also written $\frac{\mathrm{d}}{\mathrm{d}x}[fg] = f\frac{\mathrm{d}g}{\mathrm{d}x} + \frac{\mathrm{d}f}{\mathrm{d}x}g$.

132. For each function below, state whether its derivative can be found using *only* algebra, the Power Rule, the Constant Multiple Rule, and the Sum Rule. If so, give its derivative.

(a) $4x^2 - 27x$ has derivative $\boxed{8x - 27}$.

(b) $4x^2 - 27$ has derivative $\boxed{8x}$. Note that $27 = 27x^0$, so we can use the Power Rule to get $27 \cdot 0x^{-1} = 0$ as its derivative.

(c) $\sqrt{16x}$ is equal to $4x^{1/2}$ and therefore has derivative $\boxed{2x^{-1/2}}$, or $\boxed{\frac{2}{\sqrt{x}}}$.

(d) $(x + \sqrt{7})^2$ is equal to $x^2 + 2\sqrt{7}x + 7$ and therefore has derivative $\boxed{2x + 2\sqrt{7}}$.

(e) 2^{x+7} is equal to $128 \cdot 2^x$, but differentiating this requires another rule (the Exponential Rule).

(f) $\dfrac{5}{x}$ is equal to $5x^{-1}$ and therefore has derivative $\boxed{-5x^{-2}}$, or $\boxed{\frac{-5}{x^2}}$.

(g) $\dfrac{3x}{6x + 15}$ requires the Quotient Rule (or the Chain and Product Rules together).

(h) $\dfrac{6x + 15}{3x}$ is equal to $2 + \frac{5}{x}$ and therefore has derivative $\boxed{\frac{-5}{x^2}}$.

133. Using the Product Rule, give the derivative of $5^x \cdot \sin(x)$. $\boxed{5^x \cos(x) + 5^x \ln(x) \sin(x)}$

134. Use the Product Rule (twice) to find the derivative of $x^6 \cdot \cos(x) \cdot 2^x$.

$x^6 \cos(x) 2^x \ln(2) - x^6 \sin(x) 2^x + 6x^5 \cos(x) 2^x$ can be simplified to $x^5 2^x \big(x\cos(x)\ln(2) - x\sin(x) + 6$

135. True or false?

(a) $(f + g)' = f' + g'$ $\boxed{\text{true}}$

(b) $(f \cdot g)' = f' \cdot g'$ $\boxed{\text{false}}$ A correct right-hand side could be parts (c) or (d).

(c) $(f \cdot g)' = f'g + fg'$ $\boxed{\text{true}}$

(d) $\frac{\mathrm{d}}{\mathrm{d}x}(fg) = f\frac{\mathrm{d}g}{\mathrm{d}x} + g\frac{\mathrm{d}f}{\mathrm{d}x}$ true

(e) $(f \cdot g)' = g'f' + gf'$ false A correct right-hand side could be $g'f + gf'$ without the extra $'$ in the first term $g'f'$.

(f) $(f/g)' = gf' - fg'$ false A correct right-hand side could be $(gf' - fg')/g^2$.

136. Match the functions (a)-(f) to their derivatives (I)-(VI).

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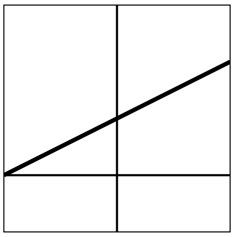

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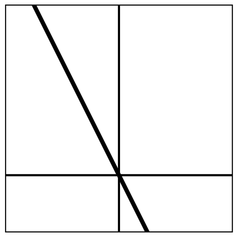

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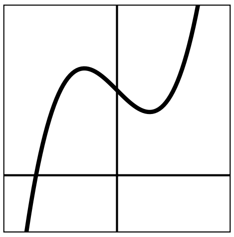

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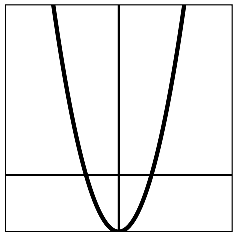

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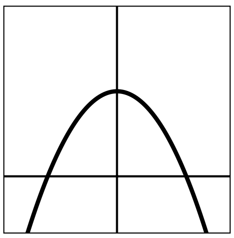

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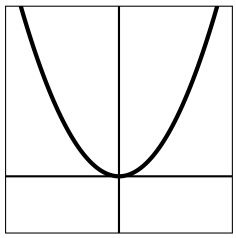

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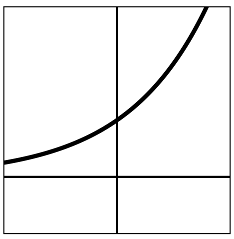

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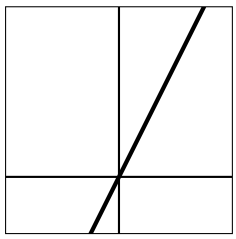

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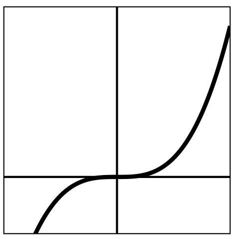

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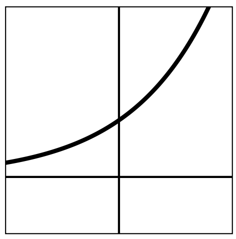

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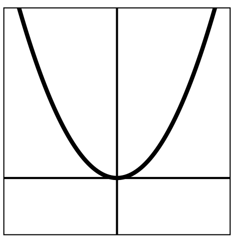

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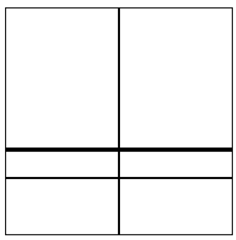

a-VI, b-II, c-I, d-V, e-III, f-IV